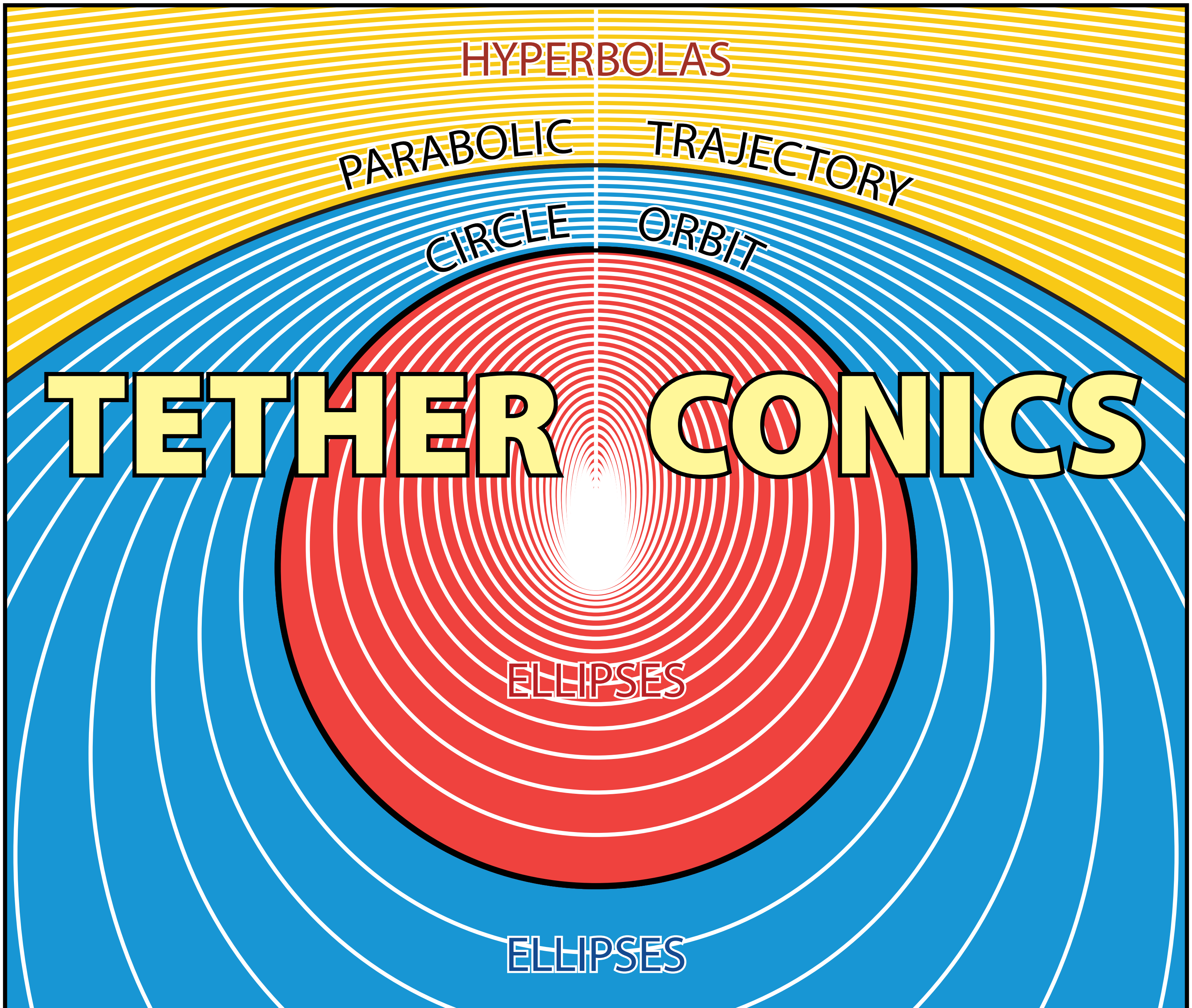
TETHER CONICS



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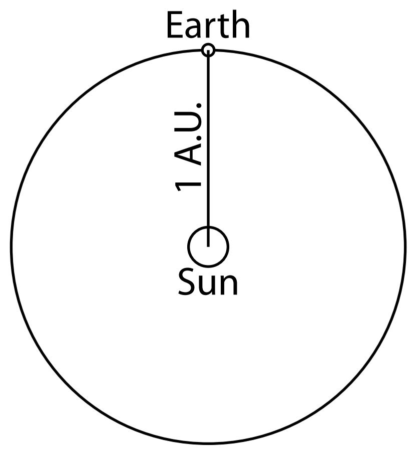
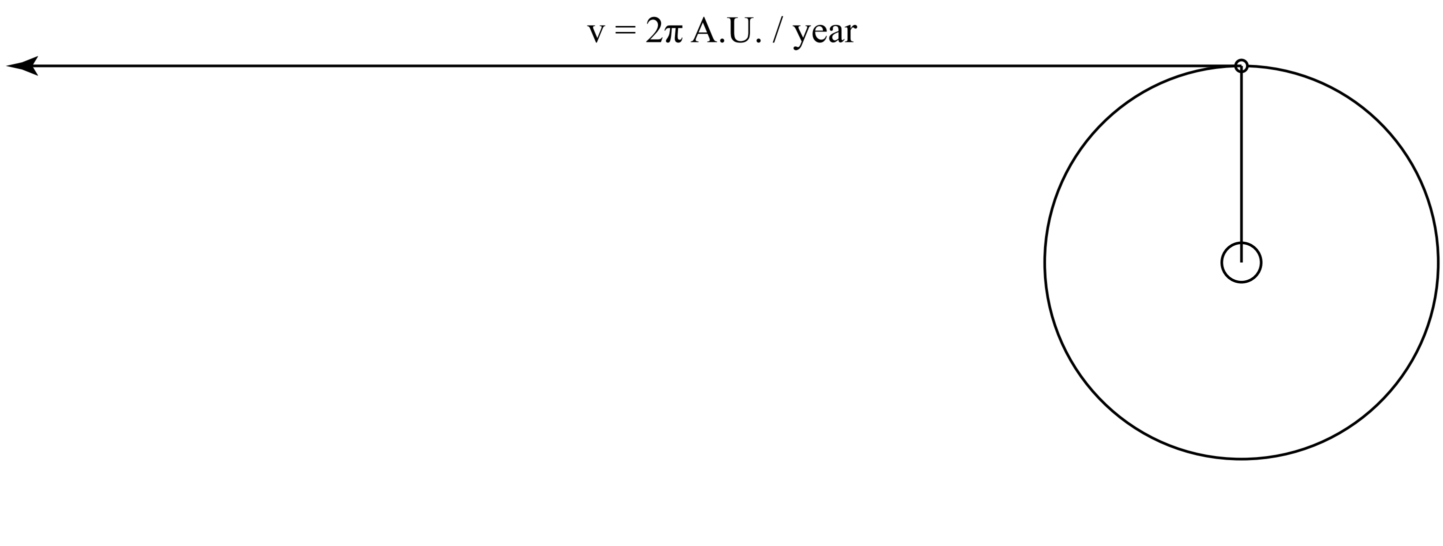
This booklet: http://cunews.info/TetherConicsBooklet.pdf



Earth's orbit has a radius of 1 A.U..
The circumference of the circular orbit is $2\ \pi$ A.U.
It takes a year to move travel over a 2π A.U. length.
Speed is 2π A.U. / year.

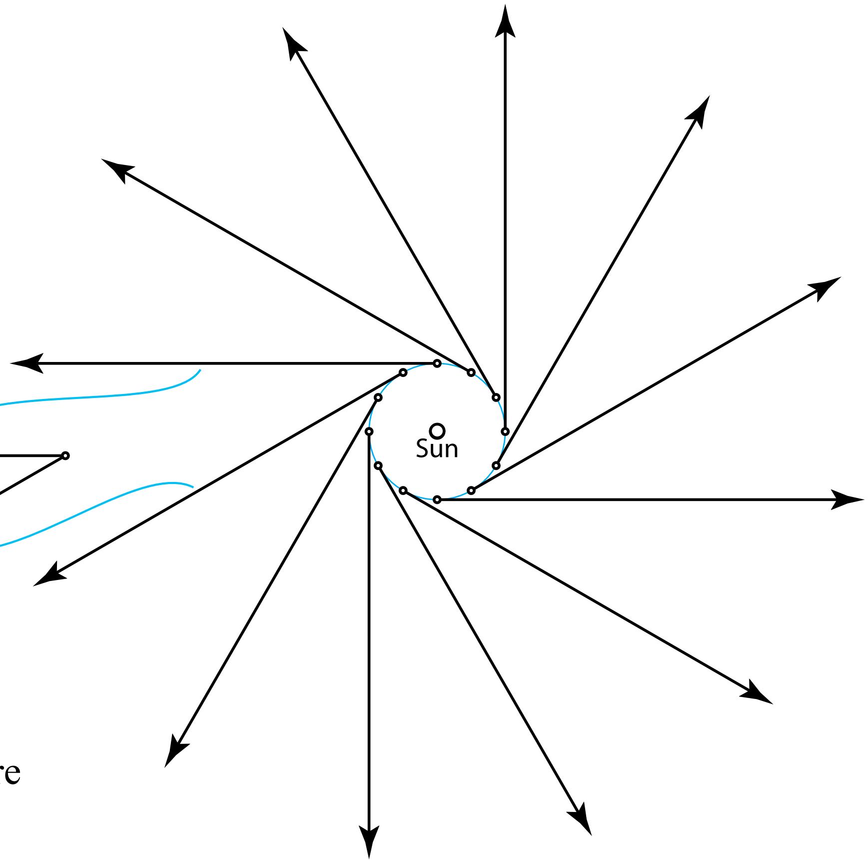
$$v = \omega r$$

where ω is angular velocity in 2π radians/(time it takes to make a circuit)

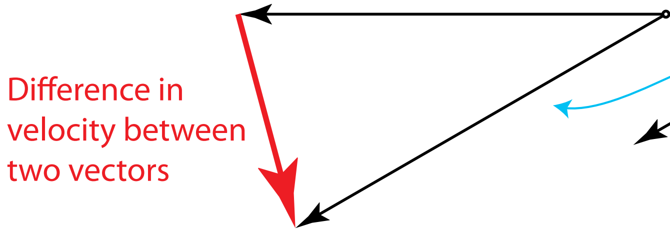


$$v = \omega r$$

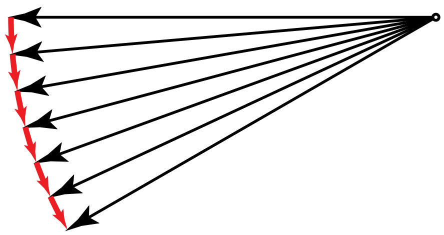
The velocity vector changes direction during the circuit around the sun.



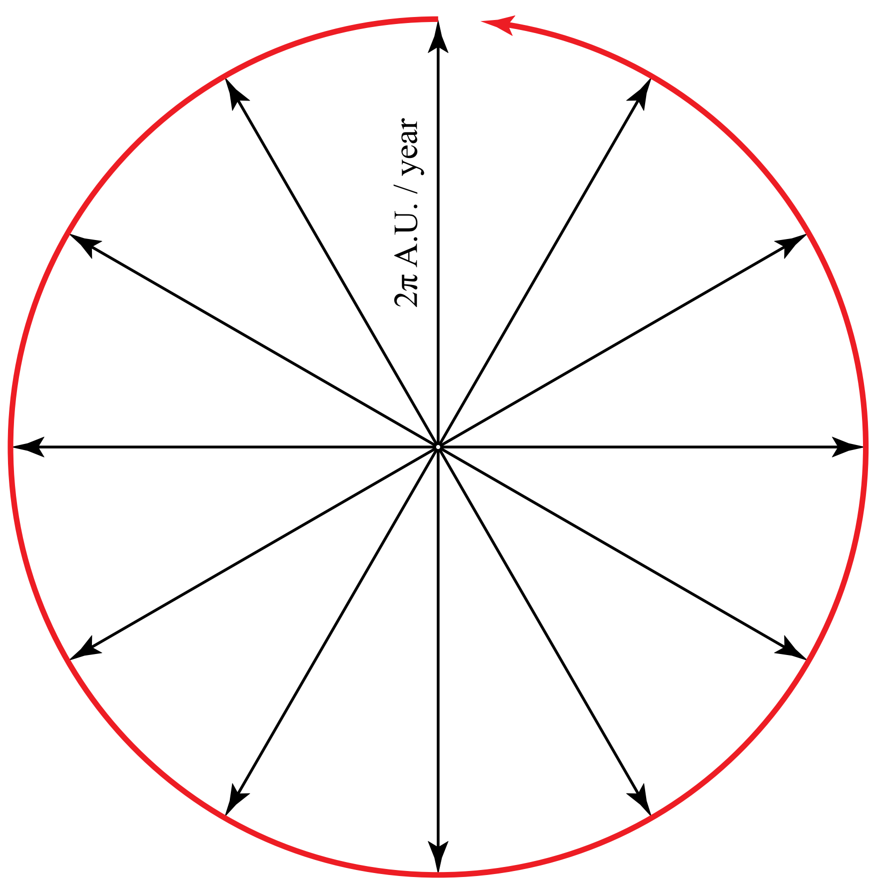
To get change of velocity from one month to the next, place the foot of one vector on the foot of another. The vector from one tip to the other is the change.



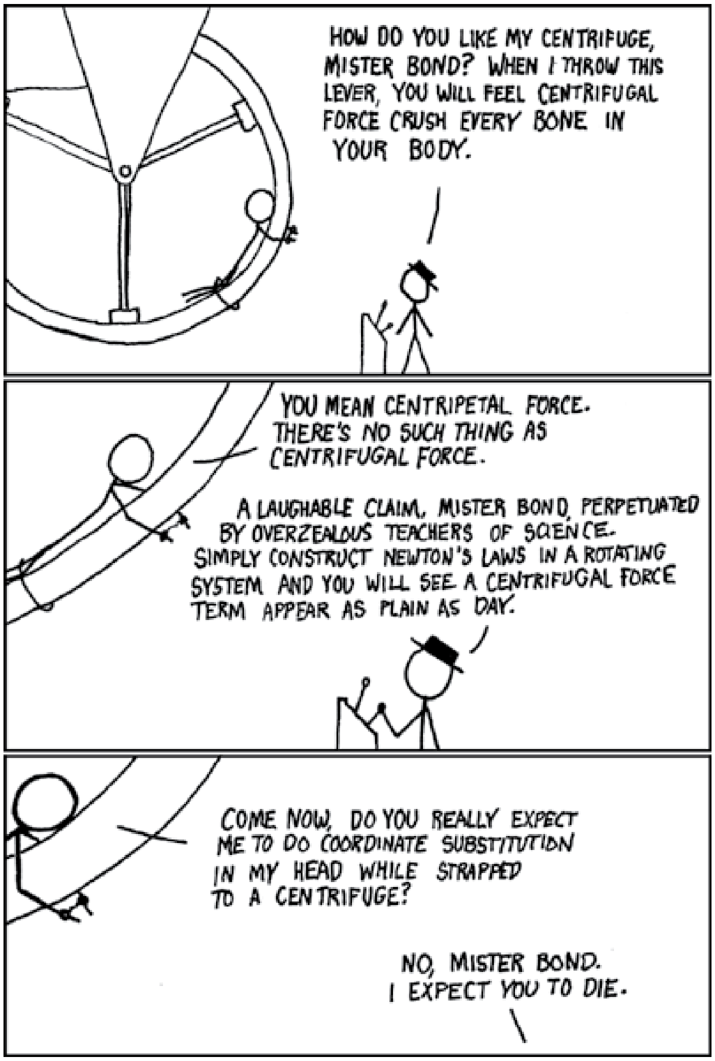
Between these two vectors there are many intermediate vectors.



Over a year's time the velocity vector traces a circle of circumference
$2\pi * v$
or 2π/year * 2π A.U. / year
$= 2\pi^2/\text{year}^2$ *A.U.
$= \omega^2 r$

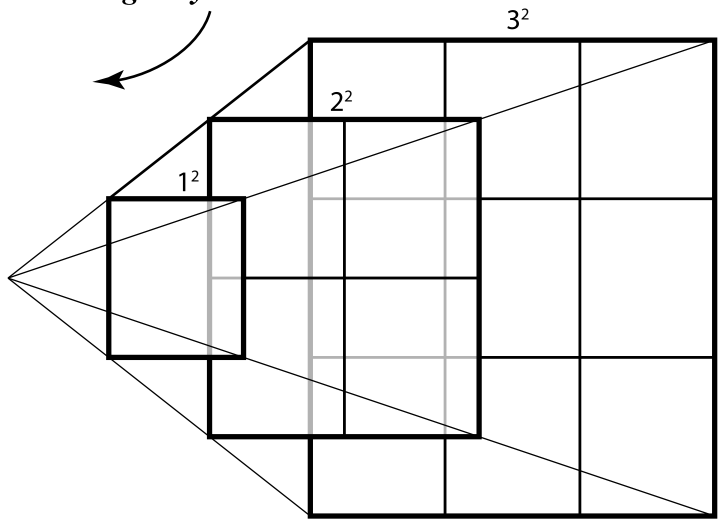


Centrifugal Acceleration = $\omega^2 r$



So centrifugal acceleration is $\omega^2 r$.

Yes, I know the so-called centrifugal force isn't really a force but inertia in a rotating frame. So to the left is the ***obligatory XKCD cartoon***.



Now, gravity falls off with inverse square of distance.

Gravity acceleration = GM / r^2.

In a circular orbit the orbiting body stays the same distance from the central gravitating body. Force of gravity cancels centrifugal force

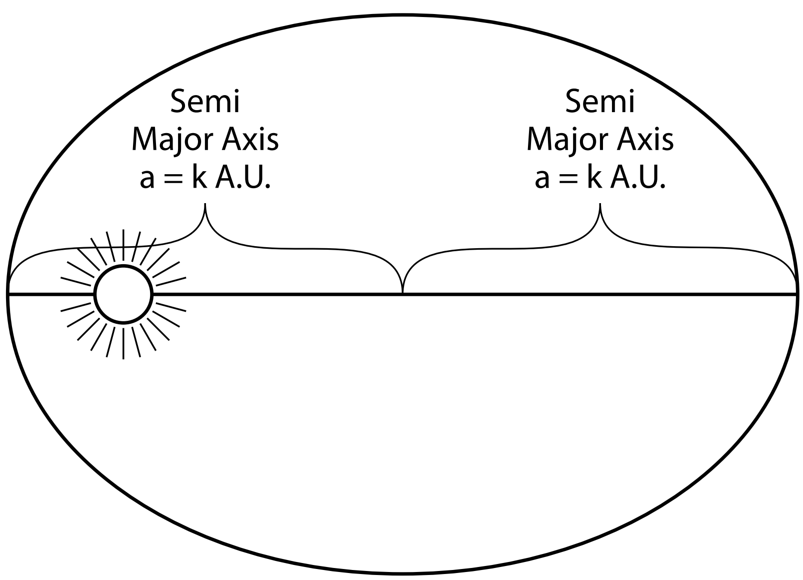
So we can say
$GM / r^2 = \omega^2 r$

$$GM = \omega^2 r^3$$

And in the case of earth's orbit about the sun, we see

$$\mathbf{GM = (2\pi / Year)^2 * A.U.^3} .$$

Kepler's Third Law



Orbital Period T is given by

$T = 2\pi\ (a^3 / GM)^{1/2}$

Where a = k A.U., k is some positive real number.

Substituting $(2\pi / \text{Year})^2 * \text{A.U.}^3$ for GM
and k A.U. for a,

$T = 2\pi\ ((k\ \text{A.U.})^3 / ((2\pi / \text{Year})^2 * \text{A.U.}^3))^{1/2}$

$T = 2\pi\ (k^3 * (\text{Year} / 2\pi\ )^2)^{1/2}$

$$\mathbf{T = k^{3/2}\ Years}$$

Does T = $k^{3/2}$ Years work? To check I went to the JPL data base and searched for Outer Main Belt Asteroids whose semi-major axi are close to 4 A.U.. What's square root of 4? 2. What's 2^3? 8. So these rocks with 4 A.U. orbits should have orbital periods of 8 years.

JPL Small-Body Database Search Engine

[Refine Search] **Results:** 42 matching objects
Constraints: orbital class (OMB) **and** a < 4.001 (au) **and** a > 3.999 (au)

object fullname ?	a (au) ?	period (years) ?
8130 Seeberg (1976 DJ1)	3.999	8
13897 Vesuvius (4216 T-2)	3.999	8
48881 (1998 HS136)	4	8
58095 Oranienstein (1973 SN)	4	8
60318 (1999 XB235)	4	8
86435 (2000 CL9)	4.001	8
121074 (1999 FA2)	4	8
145396 (2005 NE53)	4	8
185599 (2008 CJ7)	4.001	8
197528 (2004 EM19)	4.001	8
197558 (2004 FL122)	4	8
209512 (2004 RO2)	4	8
212344 (2005 UJ13)	4	8
222520 (2001 TD160)	4.001	8
239531 (2008 RK96)	3.999	8
249060 (2007 TT289)	4	8
289661 (2005 GY124)	4	8
291753 (2006 KZ5)	4	8
296860 (2009 WU225)	4	8
378566 (2008 CL205)	4	8
404653 (2014 HT42)	3.999	8
431361 (2007 DJ82)	4	8
439570 (2014 DK89)	4	8
445065 (2008 SO232)	4	8
453395 (2009 DB25)	4	8
504738 (2009 VX73)	4	8
506453 (2001 VW77)	4	8
508234 (2015 HK20)	4	8
(2003 QJ120)	4	8
(2003 UK323)	4	8
(2003 UL361)	4	8
(2006 SQ84)	3.999	8
(2010 DL89)	4	8
(2010 GF46)	4	8
(2010 KY34)	4	8
(2010 LR20)	3.999	8
(2010 LM41)	4	8
(2010 OF49)	3.999	8
(2013 ET124)	4	8
(2014 HU131)	4	8
(2014 OL202)	4	8
(2016 GM60)	3.999	8

[Refine Search]

Then I went to the same JPL data base and searched for Centaurs with close to a 9 A.U. semi major axis.

JPL Small-Body Database Search Engine

[Refine Search] **Results:** 1 matching object
Constraints: orbital class (CEN) **and** a > 8.99 (au) **and** a < 9.01 (au)

object fullname ?	a (au) ?	period (years) ?
(2011 OF45)	8.998	27

[Refine Search]

And here's some Centaurs with an approximately 16 A.U. semi-major axis.

JPL Small-Body Database Search Engine

[Refine Search] **Results:** 9 matching objects
Constraints: orbital class (CEN) **and** a > 15.8 (au) **and** a < 16.2 (au)

object fullname ?	a (au) ?	period (years) ?
10199 Chariklo (1997 CU26)	15.83	63
309139 (2006 XQ51)	15.84	63.1
(1996 AR20)	15.98	63.9
(2001 XZ255)	16	64
(2002 TK301)	16.1	64.6
(2007 RH283)	15.88	63.3
(2013 AS105)	16	64
(2014 KL84)	16.19	65.2
(2014 PQ70)	16.01	64

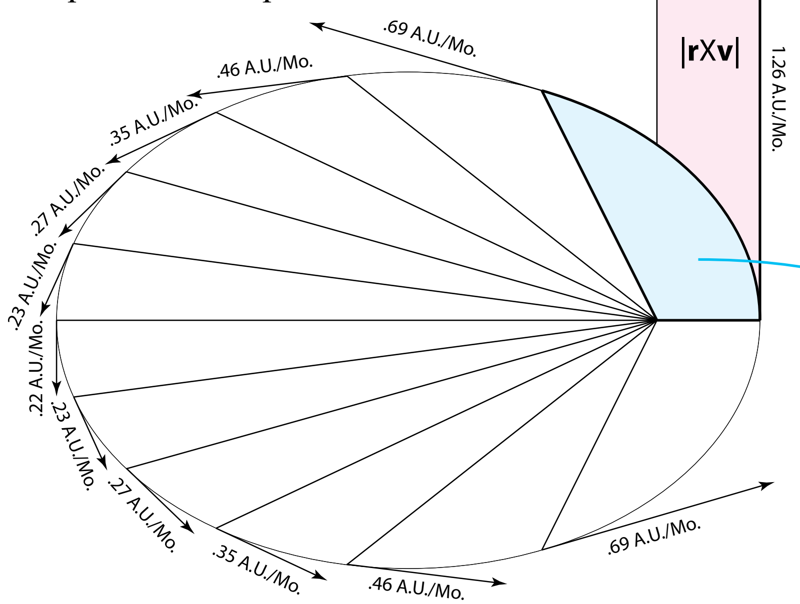
[Refine Search]

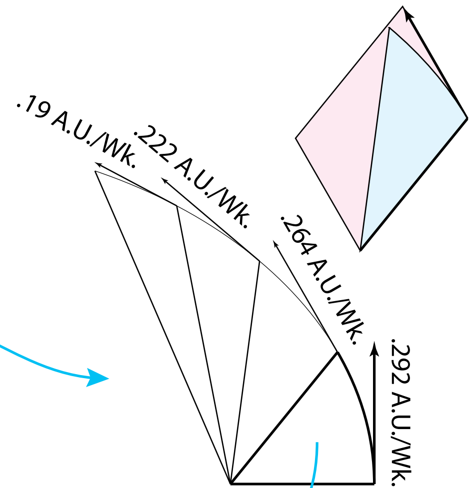
What would be the orbital period of a body with a 25 A.U. semi major axis?

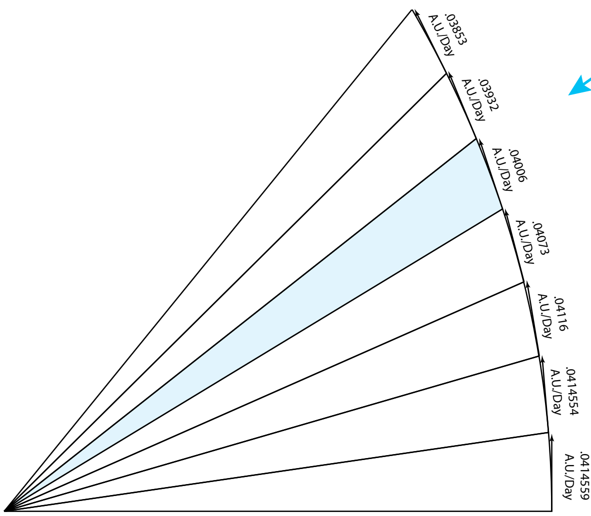
How long would it take to orbit the sun for a Kuiper Belt Object having a 36 A.U. semi major axis?

How about a scattered disc object having a 100 A.U. semi major axis?

Kepler’s 2nd Law:
Radius vector sweeps out
equal area in equal times.

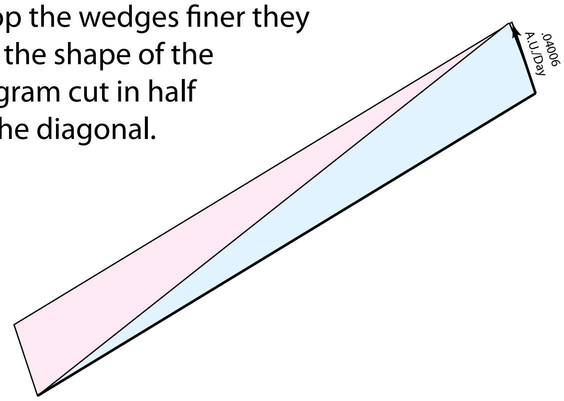






| **r** X **v** | = area of parallogram whose sides are **r** and **v**.

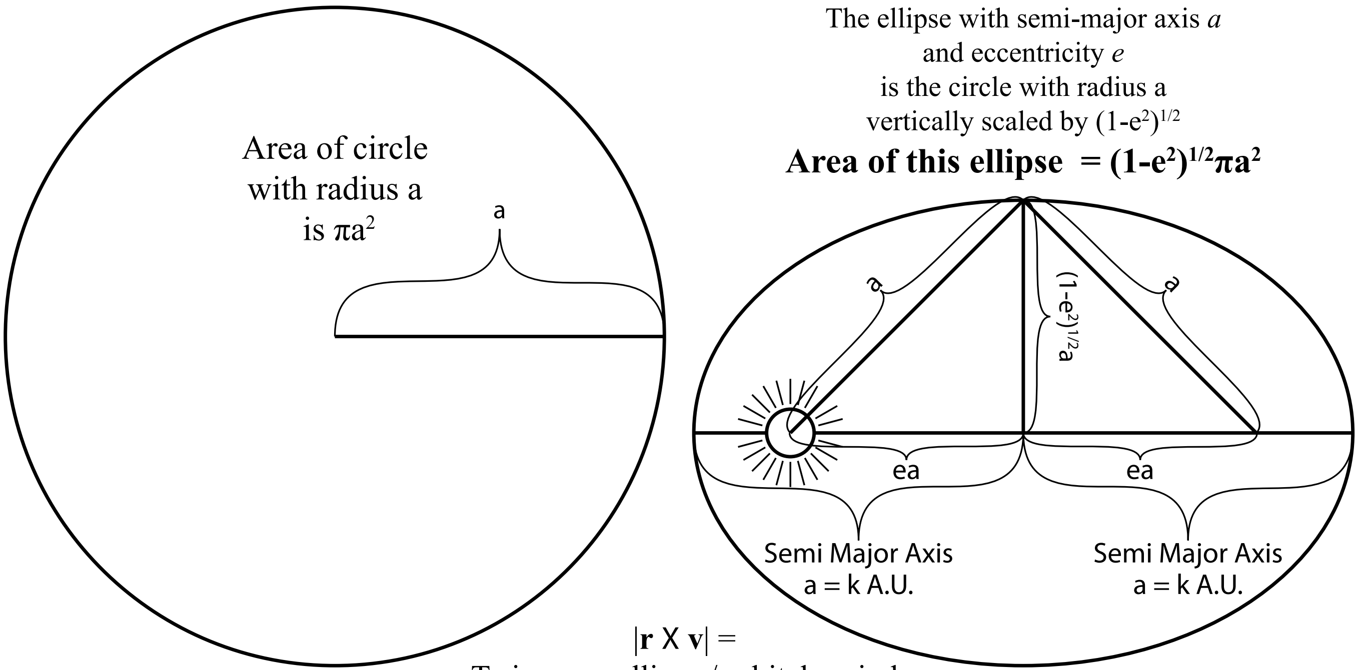
As we chop the wedges finer they approach the shape of the parallellogram cut in half through the diagonal.



Cross product of position and velocity vectors is twice the area the vector sweeps out in a given time.

The ellipse with semi-major axis *a*
and eccentricity *e*
is the circle with radius a
vertically scaled by $(1-e^2)^{1/2}$

Area of this ellipse $= (1-e^2)^{1/2}\pi a^2$



$|\mathbf{r} \times \mathbf{v}| =$
Twice area ellipse / orbital period
$= 2\ (1-e^2)^{1/2}\pi\ a^2 / T$
$= 2\ (1-e^2)^{1/2}\pi\ (k\ A.U.)^2 / (k^{3/2}\text{ years})$
$$\mathbf{= 2\ (1-e^2)^{1/2}\pi\ k^{1/2}\ A.U.^2 / year}$$

We've been using canonical units based on earth's orbit around the sun. But we can also choose canonical units based on any circular orbit around any body.

Here we'll switch gears
and base our units on
Earth's geosynchronous orbit.

We set our unit of length, R_g, to the radius of geosynchronous orbit.

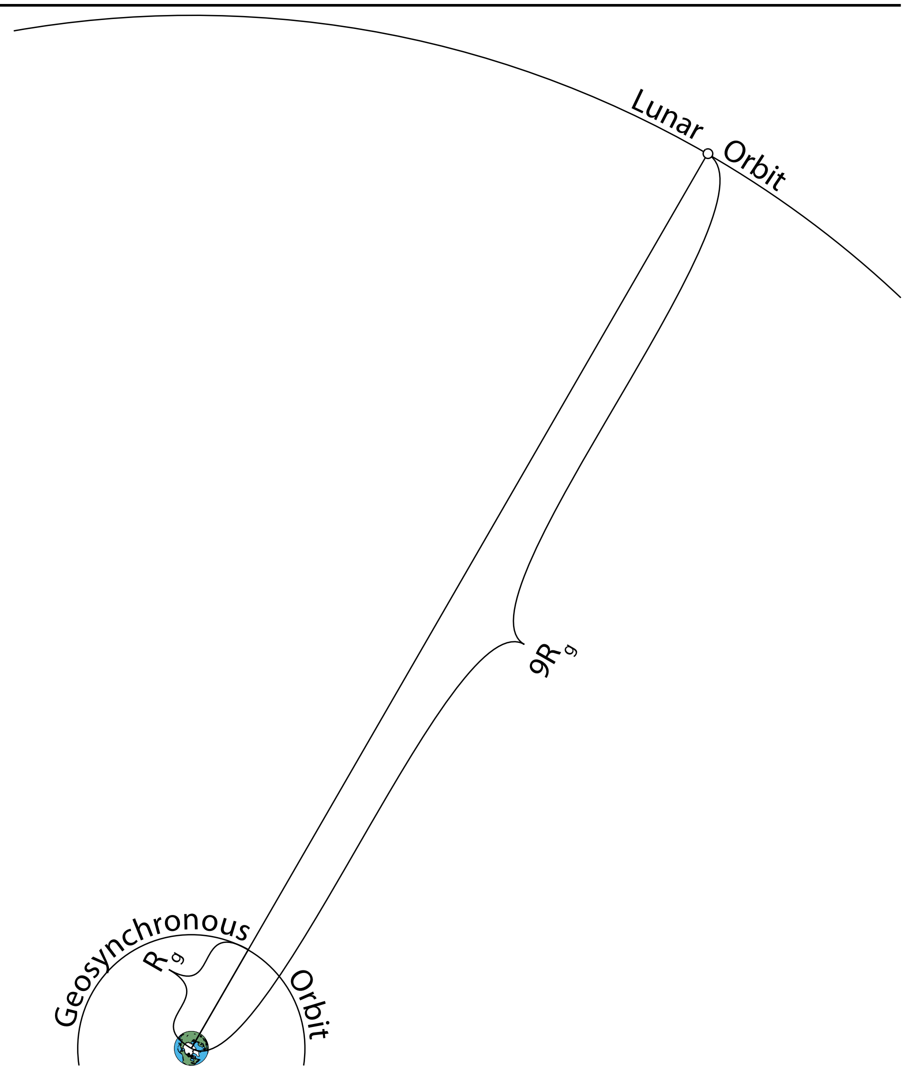
R_g = 42,300 kilometers.

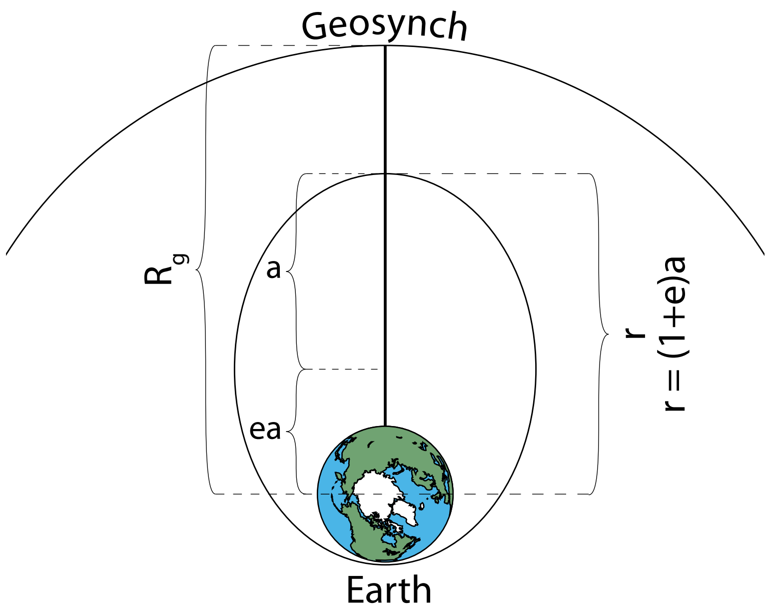
Orbital period T is one sidereal day,
T = 23 hours 56 minutes.
For this discussion
we'll just call that a day.

T = 1 day

Moon's orbital radius is 384,400 km.
384,400/42,300 = ~9.08

A lunar distance is abou 9 R_gs



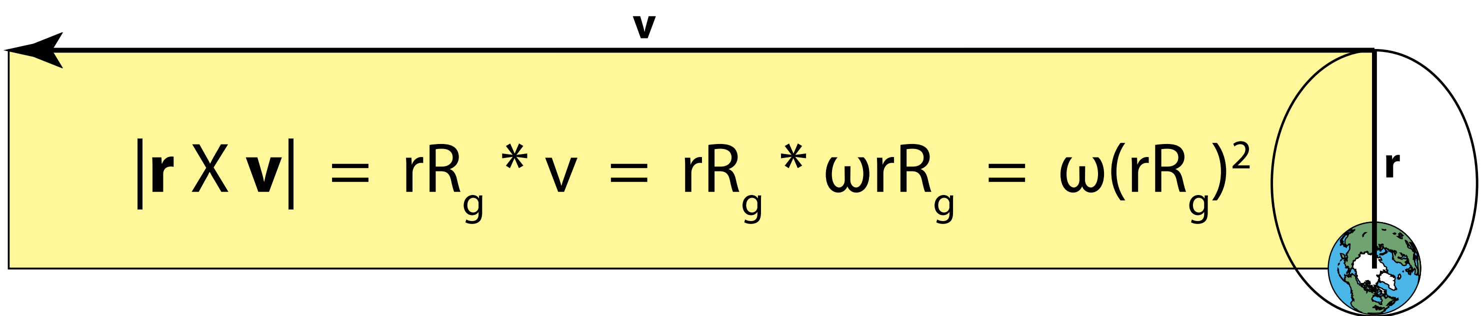


Take a point on the beanstalk.
Call the distance from this point
to earth’s center r R_g.

Note we’re using
R_g as our unit of length.

Release a payload from this point and
it will fall into an elliptical orbit with
earth’s center at a focus and
r is the apogee of this ellipse.

$$r R_g = (1+e)a$$



$$|\mathbf{r} \times \mathbf{v}| = rR_g * v = rR_g * \omega rR_g = \omega(rR_g)^2$$

Every point on the elevator is moving at the same angular velocity, 2 π radians/day.

An alert reader might say “Hey! That yellow rectangle’s area
is a lot more than twice the are of the ellipse!”

That’s because we are using a day as our time unit.
ωr would be shorter if we used T, the orbital period of this ellipse, as our time unit, .

$|\mathbf{r} \times \mathbf{v}|$ = twice ellipse area/ellipse’s orbital period

$$\omega(rR_g)^2 = (1-e^2)^{1/2} * 2\pi a^2 / T$$

Recall $a = k R_g$.

$$\omega(rR_g)^2 = (1-e^2)^{1/2} * 2\pi (kR_g)^2 / (k^{3/2} \text{ days})$$

$$2\pi/\text{day} * (rR_g)^2 = (1-e^2)^{1/2} * 2\pi k^{1/2} * R_g^{\,2} / \text{day}$$

$$(rR_g)^2 = (1-e^2)^{1/2} * k^{1/2} * R_g^{\,2}$$

$$r^2 = (k(1-e^2))^{1/2}$$

Now $rR_g = (1+e)a$ which $= (1+e)kR_g$ so $k = r/(1+e)$

$$r^2 = (r(1-e^2)/(1+e))^{1/2}$$

$$r^4 = r(1-e^2)/(1+e)$$

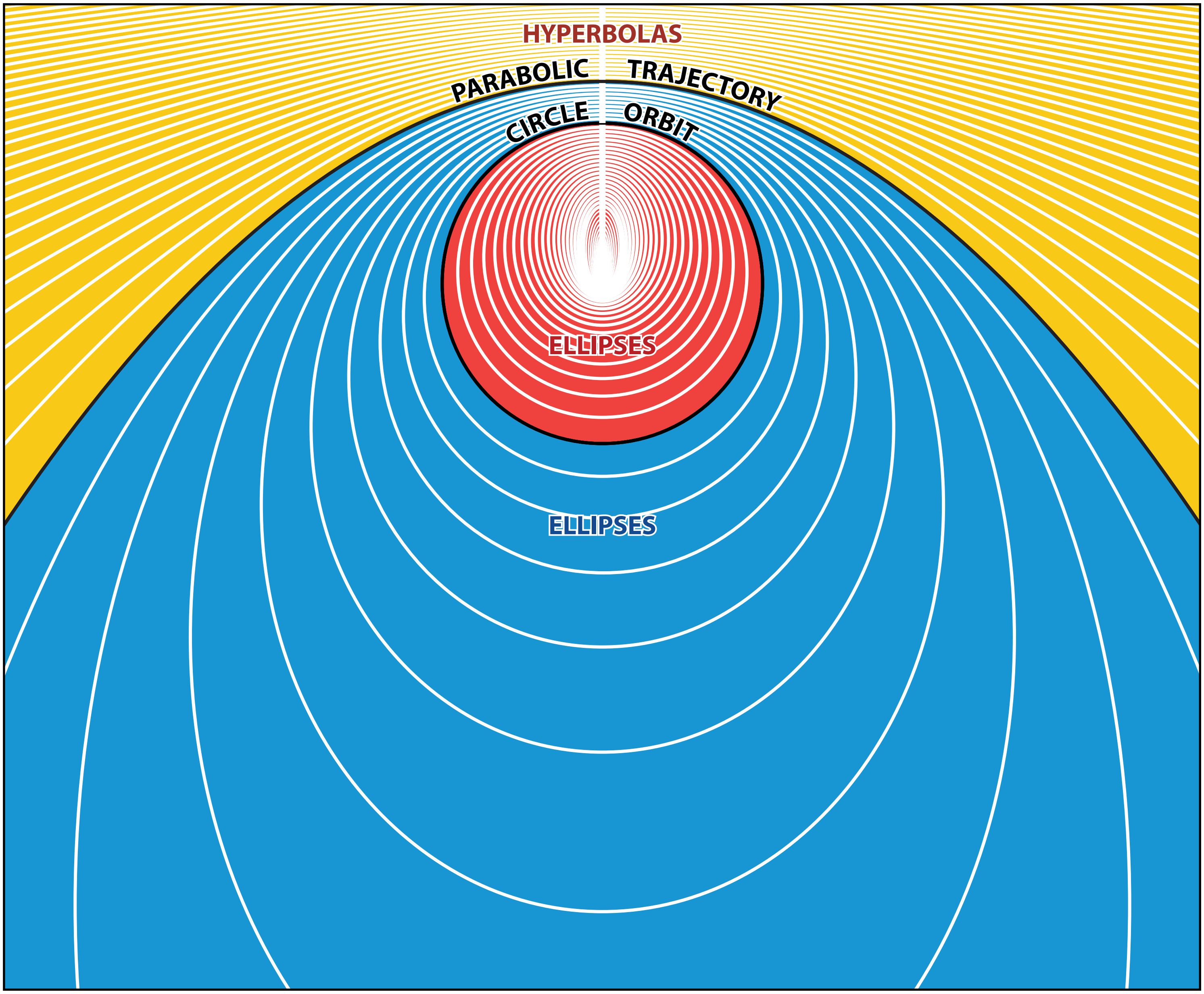
$$r^3 = 1-e$$

$$e = 1- r^3$$

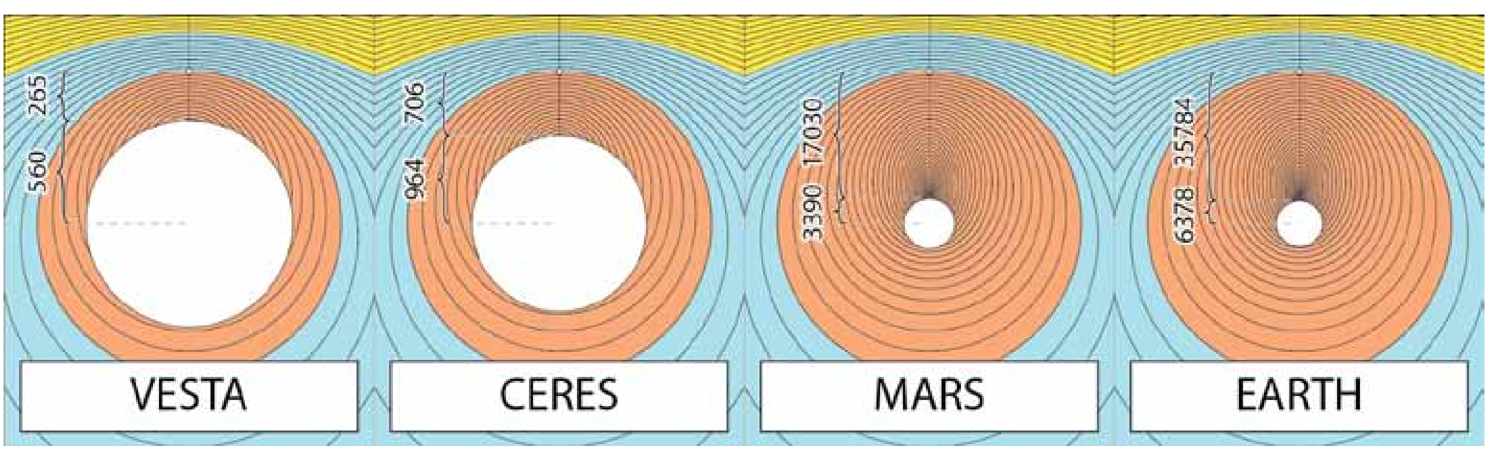
If $r > 1$, payload is released at perigee and we can use similar methods to find $e = r^3-1$.
In general

$$e = |r^3 - 1|$$

So we know the eccentricity of the conic payload follows when released from the elevator. This plus the fact that release point is at either periapsis or apoapsis of the orbit allows us to draw a family of conics associated with the elevator



Any vertical space elevator anchored to a body surface has a similar family of conics.



Above synchronous orbit centrifugal force exceeds gravity. And gravity dominates below synchronous orbit. How long does the tether above synchronous need to be to balance the length below?

This is the question P. K. Aravind looks at in his pdf *The Physics Of The Space Elevator.*
http://users.wpi.edu/~paravind/Publications/PKASpace Elevators.pdf
From Aravind's pdf:

Consider a small element of the tower of length dr whose lower end is a distance r from the Earth's center. The equilibrium of this element requires that the vector sum of the forces acting on it vanish or that $F_U+F_C-F_D-W=0$ [see Fig. 1(b)]. We write F_U-F_D as AdT, where T is the tensile stress (force per unit area) in the tower. We also use the explicit expressions for W and F_C to rewrite the equilibrium condition as

Centrifugal

$$AdT=\frac{GM(Adr\rho)}{r^2}-(Adr\rho)\omega^2 r.$$

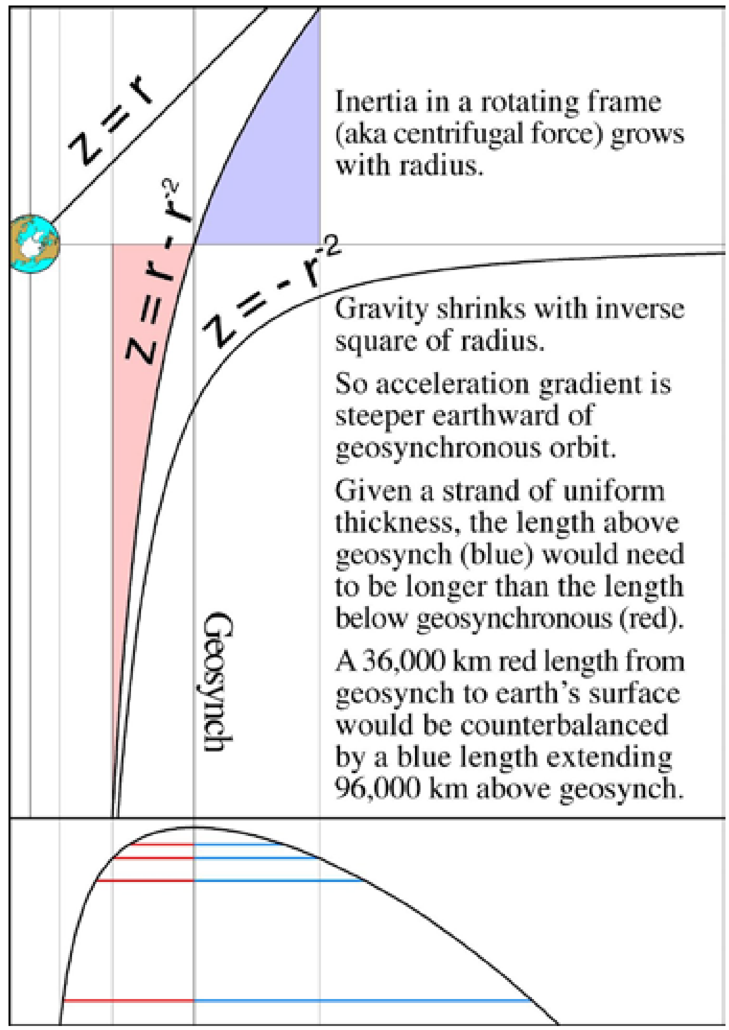
Gravity

dr

Adr = Volume tether slice.

Volume times density ρ is mass of tether slice.

Aravind integrates centrifugal acceleration and gravity along the length to get net force. To the right is an attempt to show the curve being integrated.

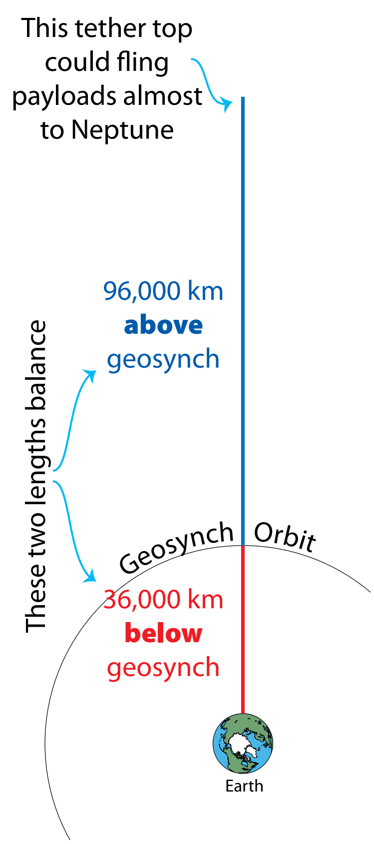


After some integrating and algebra Aravind concludes the bean stalk would need to extend 150,000 km from earth's center in order to counter-balance the length below geosynchronous orbit.

$$H=\frac{R}{2}\left[\sqrt{1+8\left(\frac{R_g}{R}\right)^3}-1\right]=150\,000\text{ km}.$$

Where
H = distance from elevator top to Earth center
R = Earth's radius, 6378 km or about .15 R_g
Rg = radius geosynchronous orbit, about 42,000 km

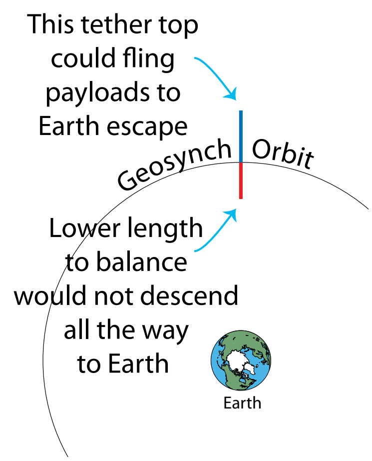
When the tether foot is at Earth's surface,
H is 150,000 kilometers.
But this elevator is implausibly large and the huge stress inflicted requires materials with extremely high tensile strength. It might be a while before we can make bucky tubes of sufficient length.



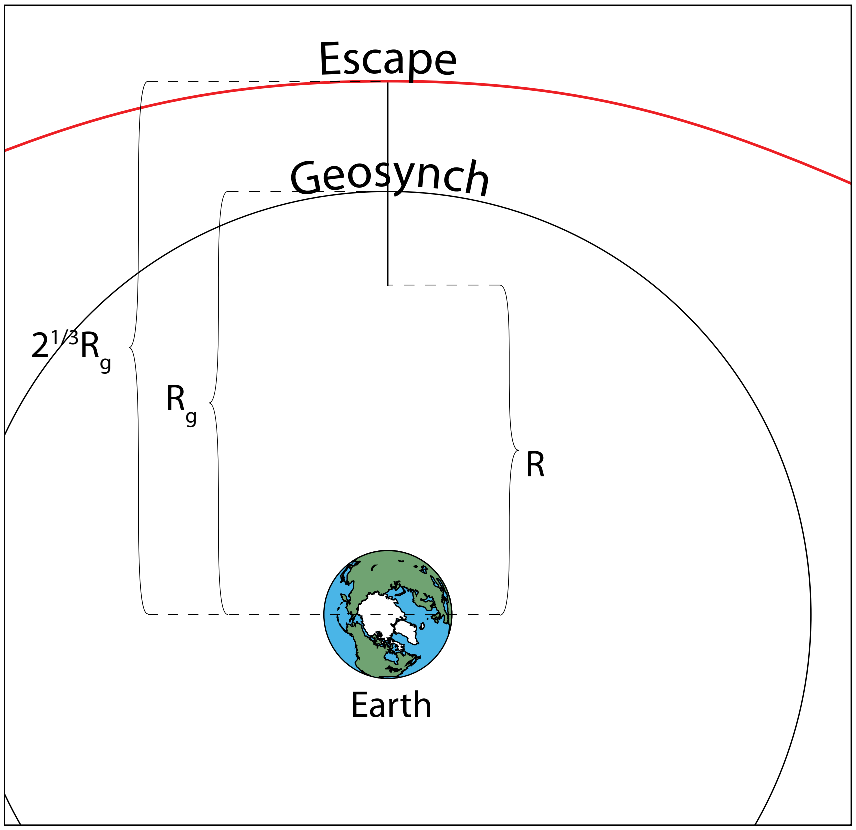
But a 142,000 km elevator isn't remotely plausible.

What about a less ambitious geosynch elevator? One capable of flinging payloads to escape.

What would the balancing lengths be?



GOLDEN TETHERS



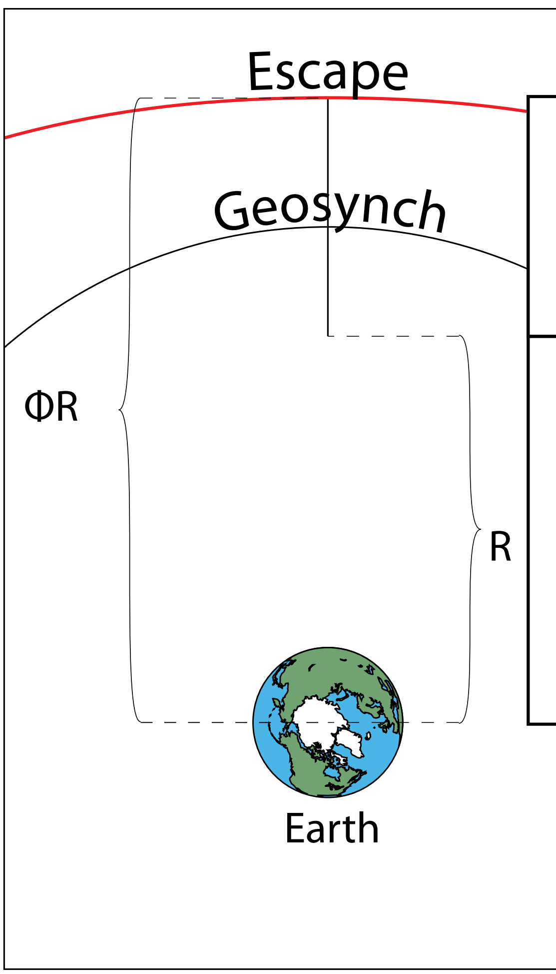
Recall $e = r^3-1$. Eccentricity parabola is 1.

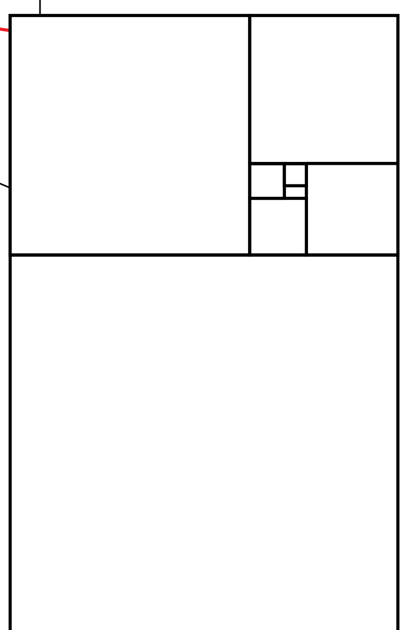
$$1 = r^3-1$$
$$r = 2^{1/3}$$
$$\text{Set } R = k\, R_g$$
$$H = R/2\ (1+8\ (R_g/R)^3)^{1/2} - 1)$$
$$2^{1/3}\ R_g = r\ R_g/2\ (1+8\ (R_g/(k\ R_g))^3)^{1/2} - 1)$$
$$2^{1/3} = k\ /2\ (1+8\ /k^3)^{1/2} - 1)$$
$$2^{1/3}\ 2/k = (1+8\ /k^3)^{1/2} - 1$$
$$\text{let } s = 2/k$$
$$2^{1/3}\ s = (1+s^3)^{1/2} - 1$$
$$2^{1/3}\ s + 1 = (1+s^3)^{1/2}$$
$$2^{2/3}\ s^2 + 2^{4/3}s + 1 = 1+s^3$$
$$s^3 - 2^{2/3}s^2 - 2^{4/3}s = 0$$
$$s^2 - 2^{2/3}s - 2^{4/3} = 0$$

Then by the quadratic formula

$$s = (2^{2/3} \pm (2^{4/3} + 4*2^{4/3})^{1/2})/2$$
$$s = 2^{2/3}(1 + 5^{1/2})/2$$
$$s = 2^{2/3}\ (1 + 5^{1/2})/2$$
$$s = 2^{2/3}\ \Phi$$
$$2/k = 2^{2/3}\ \Phi$$

$$\Phi k = 2^{1/3}$$



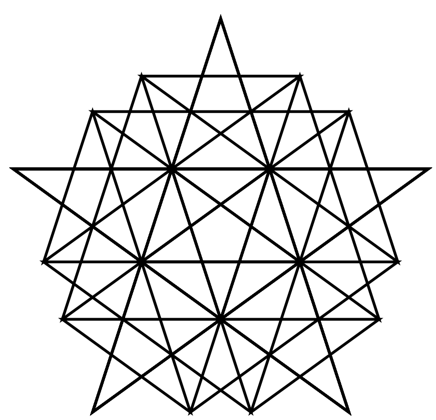


It's our old friend the

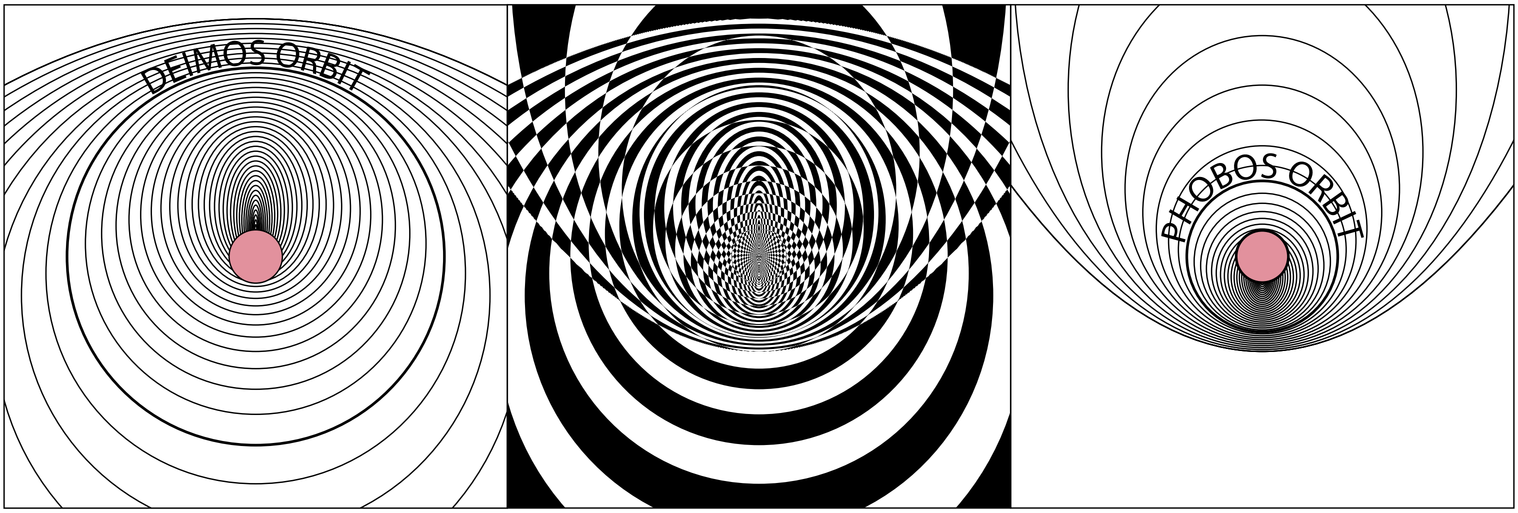
Golden Mean!

Given a vertical tether whose top can hurl payloads at escape velocity and a lower tether length that balances the upper length, radius tether top is 1.618 the radius of tether foot.

It was a pleasant surprise to see this number pop up when I didn't expect it.



Z R V T O
Zero Relative Velocity Transfer Orbit



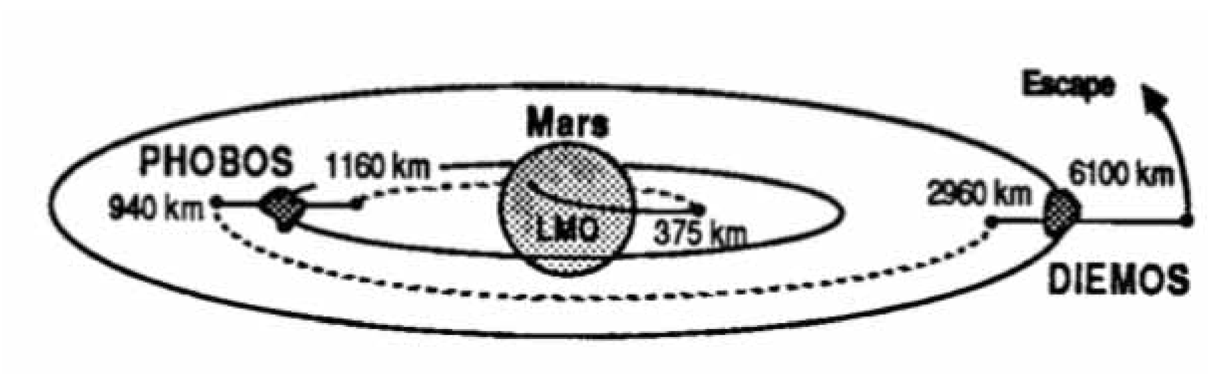
Anchor a vertical elevator on Deimos.
Between Deimos circular orbit and Mars' center there are ellipses of every eccentricty between 0 and 1.

Anchor an elevator at Phobos.
Between Phobos circular orbit and the parabola there are also ellipses of every eccentricity between 0 and 1.

Do the Phobos and Deimos elevators share an ellipse?
Overlapping the two families of conics, the moiré pattern seems to indicate a shared ellipse.

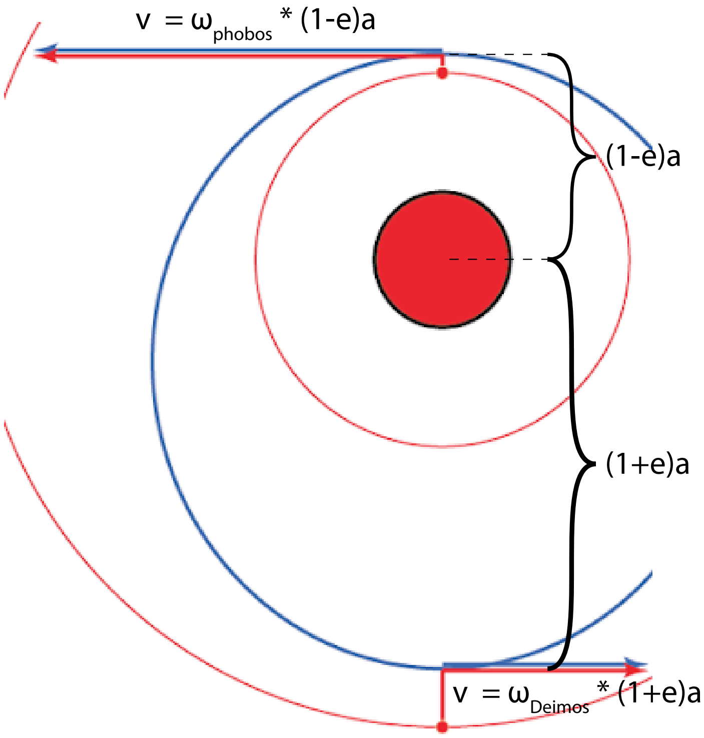
At periapsis a payload traveling along this elliptical orbit would have the same relative velocity as the rendezvous point on a Phobos elevator. At apoapsis the payload would have the same relative velocity as the rendezvous point on a Deimos tether.

Using this **Zero Relative Velocity Transfer Orbit** the two moons could exchange payloads using virtually zero reaction mass.



Paul Penzo, a JPL engineer, talked about this possible path between Deimos and Phobos elevators back in 1984. Above is Penzo's illustration from that paper.

I believe ZRVTO is a term coined by Marshall Eubanks who is also an advocate of PAMSE -- Phobos Anchored Mars Space Elevator.



The top of the Phobos tether is moving the same angular velocity as Phobos, ω_{phobos}.

The bottom of the Deimos tether is moving the same angular velocity as Deimos, ω_{Deimos}.

$$\text{Specific angmom} = \mathbf{v}_{periaerion} \times \mathbf{r}_{periaerion}$$
$$\text{Specific angmom} = \mathbf{v}_{apoiaerion} \times \mathbf{r}_{apoiaerion}$$
$$\mathbf{v}_{periaerion} \times \mathbf{r}_{periaerion} = \mathbf{v}_{apoiaerion} \times \mathbf{r}_{apoiaerion}$$
$$\omega_{Phobos} * ((1-e)a)^2 = \omega_{Deimos} * ((1+e)a)^2$$

$$\boxed{e = (1 - (\omega_{Deimos}/\omega_{Phobos})^{1/2})/(1 + (\omega_{Deimos}/\omega_{Phobos})^{1/2})}$$

$$\text{Specific angmom} = \omega_{Phobos} * r^2 = (a(1-e^2)\mu)^{1/2}$$

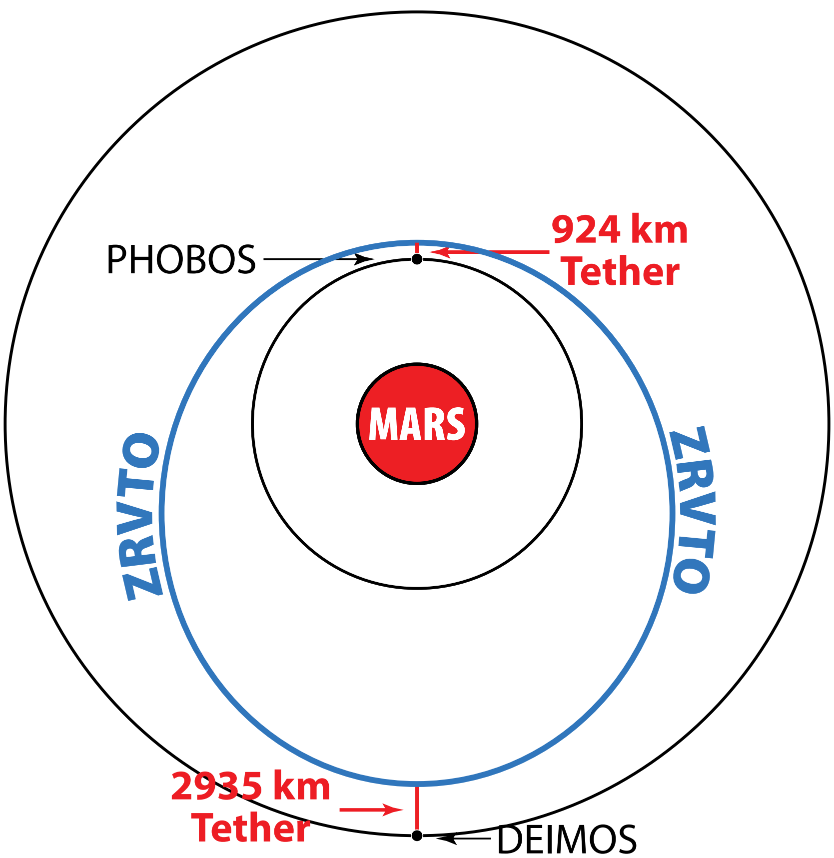
At periapsis r is (1-e)a. So a = r/(1-e). Substituting:

$$\omega_{Phobos} * r^2 = (r(1+e)\mu)^{1/2}$$
$$r^4 = r(1+e)\mu/\omega_{Phobos}{}^2$$
$$r = ((1+e)\mu/\omega_{Phobos}{}^2)^{1/3}$$

$$\boxed{r_{periaerion} = (1 + e)^{1/3} r_{Phobos}}$$

Similarly:

$$\boxed{r_{apoaerion} = (1 - e)^{1/3} r_{Deimos}}$$



Angular velocities as well as orbital radii of Phobos and Deimos are easily found on Wikipedia.

Plugging these into the above equations we find an ~1000 km tether ascending from Phobos and a ~3000 km tether descending from Deimos is sufficient for a ZRVTO route between the two moon.

It is reassuring my numbers are close to Paul Penzo's. Wnile I'm a rank amateur, Penzo was an accomplished aerospace engineer.

Not just Phobos & Deimos

This technique can be used for any pair of tidelocked moons in nearly circular, coplanar orbits.

A bunch of moons orbiting our gas giants fit this description.

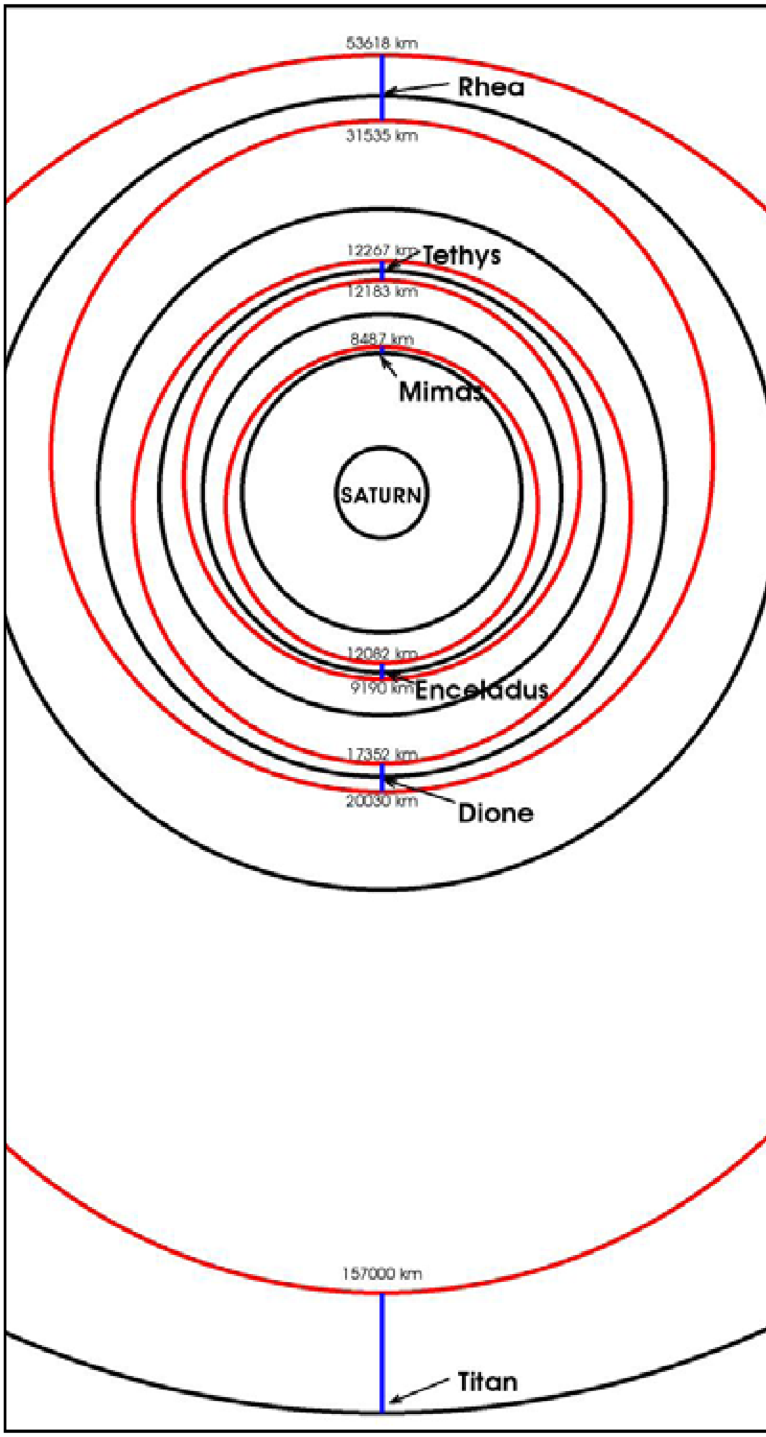
In the following page I look at Saturn's and Jupiter's moons.

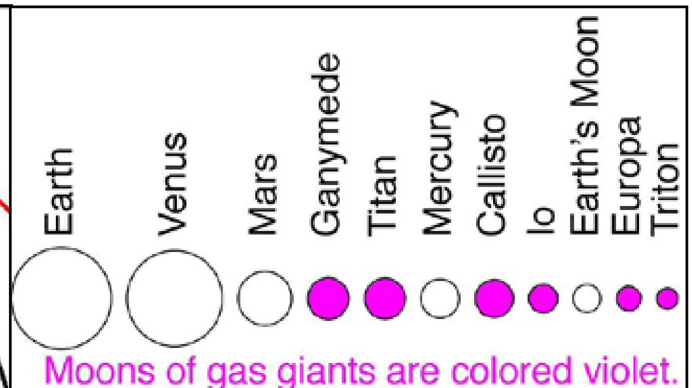
Fast paced science fiction set in our solar system is implausible. Trip times and launch windows are on the order of months and years in the inner solar system, years and decades in the outer solar system.

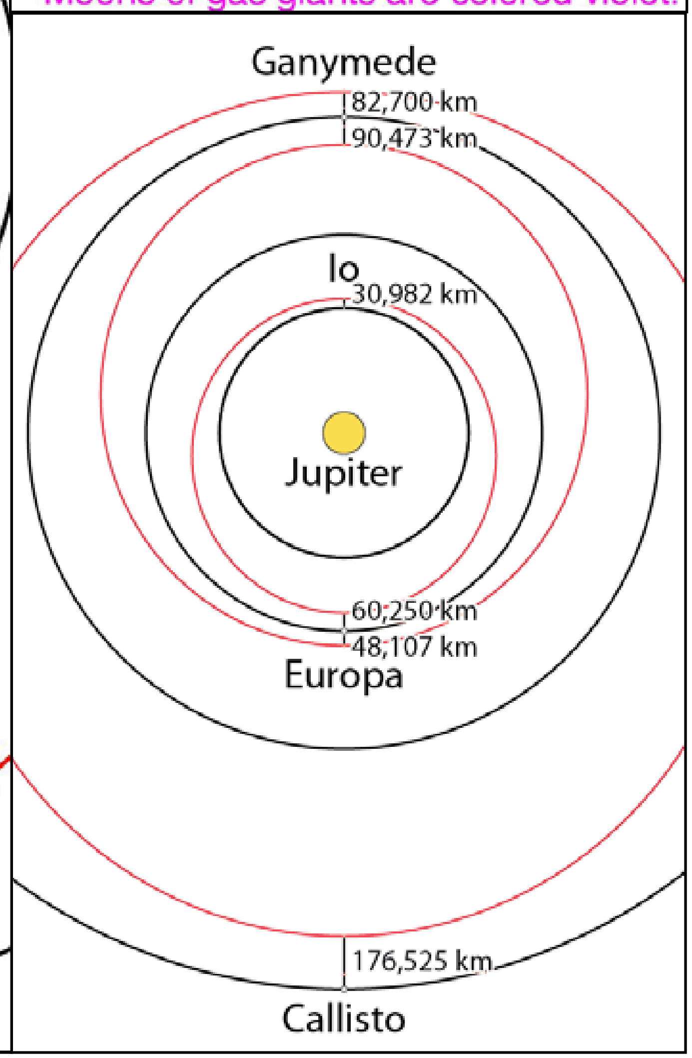
It's a different story in a gas giant's family of moons. Trip time and launch windows are on the order of days and weeks.

These are tide locked moons with nearly circular coplanar orbits. So ZRVTO routes between moon elevators might be possible.

There might be gas giants with families of large moons in other star systems as well. A gas giant in the "Goldilocks Zone" with a family of large moons would be a great science fiction setting.







Thanks!

To folks in Facebook groups who help proofread this booklet and offer suggestions: Greg Bullock, MolbOrg Ogrovitch, Keith Blockus and Duffy Toler. Also folks in the Math Stack Exchange: Yves Daoust, robjohn and Claude Leibovici.

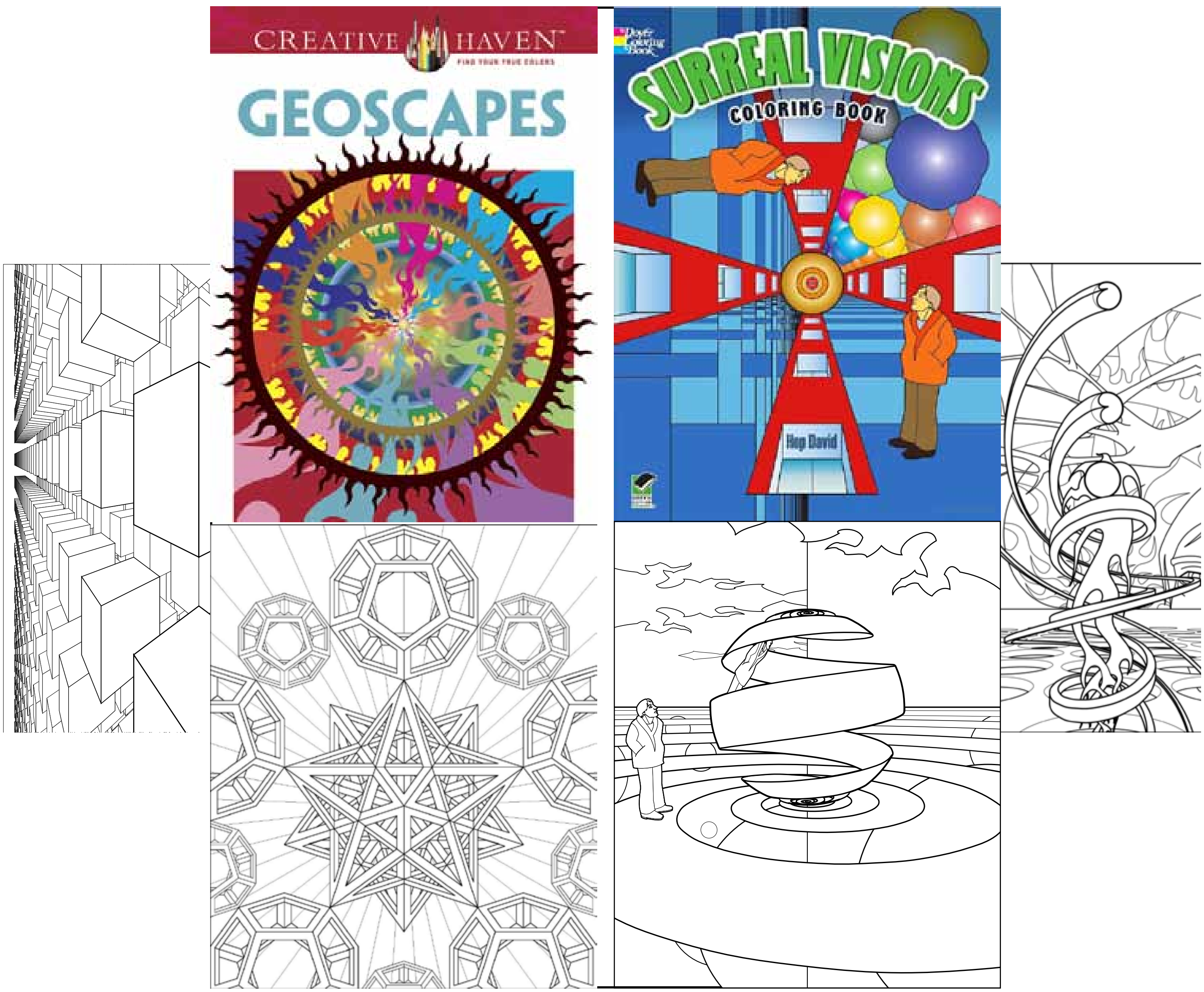
A few blog posts on orbital tethers and space elevators:

Phobos, Panama Canal Inner Solar System: *http://hopsblog-hop.blogspot.com/2015/06/phobos-panama-canal-of-inner-solar.html*
Deimos Tether: *http://hopsblog-hop.blogspot.com/2016/01/deimos-tether.html*
Upper Phobos Tether: *http://hopsblog-hop.blogspot.com/2016/01/deimos-tether.html*
Lower Phobos Tether: *http://hopsblog-hop.blogspot.com/2015/12/lower-phobos-tether.html*
Pluto Charon Elevator: *http://hopsblog-hop.blogspot.com/2016/08/pluto-charon-elevator.html*
Wolfe's Tether Spreadsheet: *http://hopsblog-hop.blogspot.com/2015/12/how-wolfes-tether-spreadsheet-works.html*
Transcislunar Railroad: *http://hopsblog-hop.blogspot.com/2016/08/tran-cislunar-railroad.html*
Mini Solar Systems: *http://hopsblog-hop.blogspot.com/2013/01/mini-solar-systems.html*
Orbital Momentum As A Commodity *http://hopsblog-hop.blogspot.com/2015/05/orbital-momentum-as-commodity.html*

Hop's Deviant Art Gallery: *https://hop41.deviantart.com*

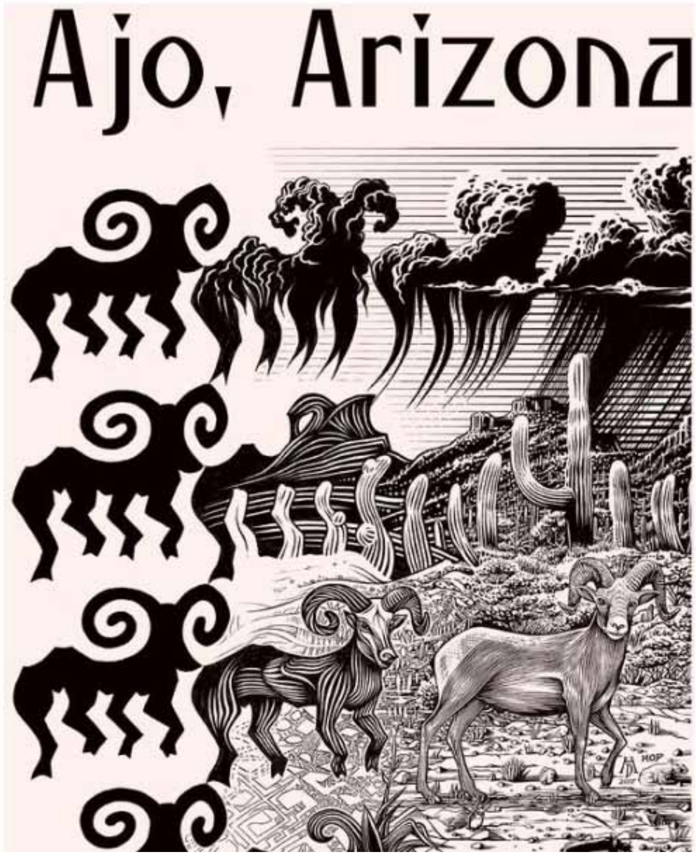
Coloring Books by Hop David

Conic Sections and Celestial Mechanics Coloring Book Out of print but pdf available: *http://clowder.net/hop/TMI/pages1-40.pdf*
Geoscapes, Surreal Visions

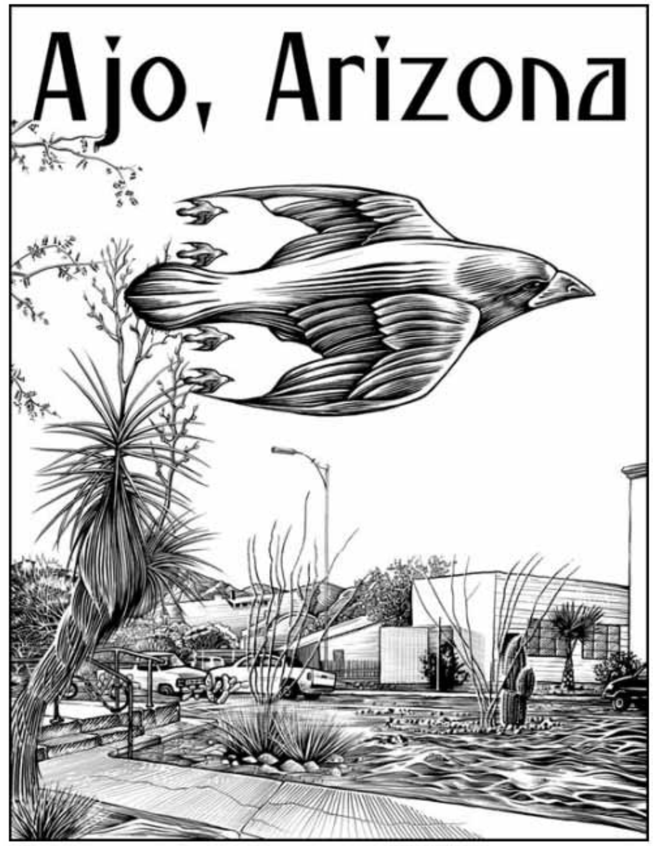


Hop T-Shirts on Ajo-Copper-News.com

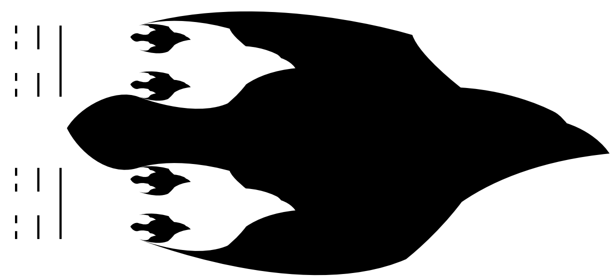
Ajo Tessellation Shirt



Ajo How Many Birds? Shirt



Puerto Peñasco Shirt



I call these ***Cantor Birds***. Draw a line from wing tip to wing tip and the tail feathers punches out the middle third of the line segment. The next iteration of birds puches out the middle thirds of remaining line segments. And so on.

Fractal fish tessellation in Puerto Peñasco shirt is by Robert Fathauer.